OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS **2604**

Pure Mathematics 4

Wednesday **12 JANUARY 2005** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 3 printed pages and 1 blank page.

1 A curve has equation $y = \dfrac{18x - 5x^2}{x^2 - 36}$.

(i) Write down the equations of the three asymptotes. [3]

(ii) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. Hence find the coordinates of the stationary points. [6]

(iii) Sketch the curve. [5]

(iv) On a separate diagram, sketch the curve with equation $y = \left|\dfrac{18x - 5x^2}{x^2 - 36}\right|$. [3]

(v) State the values of k for which the equation $\left|\dfrac{18x - 5x^2}{x^2 - 36}\right| = k$ has exactly three distinct real solutions. [3]

2 **(a)** Find the sum of the series

$$(1 \times 7) + (3 \times 11) + (5 \times 15) + \ldots + (2n - 1)(4n + 3),$$

giving your answer in a fully factorised form. [6]

(b) Solve the inequality $\dfrac{x}{x-1} < \dfrac{x-1}{x}$. [6]

(c) Express $\dfrac{9r + 14}{r(r+1)(r+2)}$ in partial fractions, and hence find the sum of the first n terms of the series

$$\frac{23}{1 \times 2 \times 3} + \frac{32}{2 \times 3 \times 4} + \frac{41}{3 \times 4 \times 5} + \ldots .$$ [8]

3 Throughout this question, $\alpha = 3 + 2\text{j}$.

(a) **(i)** Find α^2 and α^3. [3]

(ii) Given that α is a root of the equation $2x^3 + px^2 + 20x + q = 0$, where p and q are real numbers,

(A) find p and q, [5]

(B) find the other two roots of the cubic equation. [4]

(b) **(i)** Find $|\alpha|$ and $\arg \alpha$. [2]

(ii) On an Argand diagram, shade the region corresponding to complex numbers z for which

$$|z - \alpha| \leqslant 2.$$ [2]

(iii) Given that $|z - \alpha| \leqslant 2$, find

(A) the minimum possible value of $|z|$,

(B) the maximum possible value of $|z|$,

(C) the maximum possible value of $\arg z$. [4]

4 **(a)** Given that $\mathbf{M} = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$, prove by induction that $\mathbf{M}^n = \begin{pmatrix} 1-3n & 9n \\ -n & 1+3n \end{pmatrix}$, where n is a positive integer. [7]

(b) **(i)** Find the vector product $(2\mathbf{i} - 9\mathbf{j} - 8\mathbf{k}) \times (5\mathbf{i} + 10\mathbf{j} + 6\mathbf{k})$. [2]

(ii) Find the equation of the line of intersection of the two planes

$$\begin{aligned} 2x - 9y - 8z &= 48, \\ 5x + 10y + 6z &= -10. \end{aligned}$$ [3]

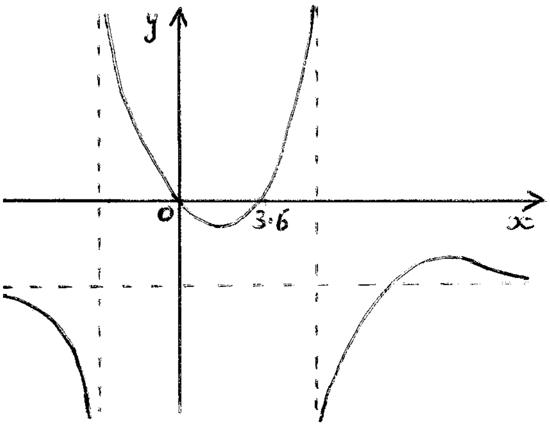
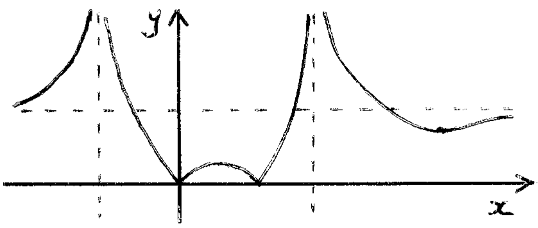
(iii) Given that $\begin{pmatrix} 2 & -9 & -8 \\ 5 & 10 & 6 \\ 2 & 1 & k \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 48 \\ -10 \\ 13 \end{pmatrix}$, and $k \neq 0$, express x, y and z in terms of k. [6]

(iv) Describe geometrically how the following three planes intersect.

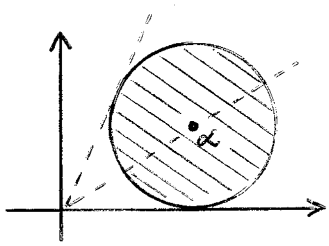
$$\begin{aligned} 2x - 9y - 8z &= 48 \\ 5x + 10y + 6z &= -10 \\ 2x + y \phantom{{}+6z} &= 13 \end{aligned}$$ [2]

Mark Scheme

MARKING SCHEME

1 (i)	$x=6,\ x=-6,\ y=-5$	B1B1B1 **3**	*Deduct 1 if there are any extras*
(ii)	$\frac{dy}{dx}=\frac{(x^2-36)(18-10x)-(18x-5x^2)(2x)}{(x^2-36)^2}$	M1 A1	Use of quotient rule (or equivalent) with at most one error Any correct form
	$=\frac{-18(x^2-20x+36)}{(x^2-36)^2}$		
	$=0$ when $x^2-20x+36=0$	M1	Using $\frac{dy}{dx}=0$ to obtain quadratic eqn
	$x=2,\ 18$	M1	Solving quadratic
	Stationary points are $(2,\ -\frac{1}{2}),\ (18,\ -\frac{9}{2})$	A1A1 **6**	If A0, give A1 for $x=2,\ 18$
(iii)		B1 B1 B1 B1 B1 **5**	LH section: negative gradient, below x-axis Middle section: minimum to right of y-axis Crossing x-axis at 0, 3.6 Maximum in RH section Approaching asymptotes correctly
(iv)		B1 ft B1 ft B1 **3**	At least one negative section reflected Fully correct general shape *(B0 if graph in (iii) has a section missing)* Two sharp points on x-axis
(v)	$k=\frac{1}{2},\ \frac{9}{2},\ 5$	B1B1B1 ft **3**	Only isolated values can score e.g $\frac{9}{2}\le k\le 5$ scores B0 ft only for exactly equivalent work *Max B2 if there are any extras*

2 (a)	$\sum_{r=1}^{n}(8r^2+2r-3)$ $=\frac{8}{6}n(n+1)(2n+1)+n(n+1)-3n$ $=\frac{1}{3}n(8n^2+15n-2)$ $=\frac{1}{3}n(n+2)(8n-1)$	B1 B1B1B1 ft M1 A1 **6**	for $8r^2+2r-3$ Collecting terms
(b)	$\frac{x}{x-1}=\frac{x-1}{x}$ when $x=\frac{1}{2}$ $\frac{x}{x-1}<\frac{x-1}{x}$ when $x<0,\ \frac{1}{2}<x<1$	M1 B1 M2 A1A1 **6**	Solving $\frac{x}{x-1}=\frac{x-1}{x}$ or $\frac{x}{x-1}-\frac{x-1}{x}<0$ or $x^3(x-1)<x(x-1)^3$ for $x=\frac{1}{2}$ or factor $(2x-1)$ Considering intervals defined by three critical values $\frac{1}{2}$ (ft), 0, 1 Give M1 for $\frac{1}{2}$ and either 0 or 1 *Dependent on M2* *Correct answer always earns 6 marks*
(c)	$\frac{7}{r}-\frac{5}{r+1}-\frac{2}{r+2}$ Sum $=\left(\frac{7}{1}-\frac{5}{2}-\frac{2}{3}\right)+\left(\frac{7}{2}-\frac{5}{3}-\frac{2}{4}\right)+\ldots$ $+\left(\frac{7}{n-1}-\frac{5}{n}-\frac{2}{n+1}\right)+\left(\frac{7}{n}-\frac{5}{n+1}-\frac{2}{n+2}\right)$ $=\frac{7}{1}-\frac{5}{2}+\frac{7}{2}-\frac{2}{n+1}-\frac{5}{n+1}-\frac{2}{n+2}$ $=8-\frac{7}{n+1}-\frac{2}{n+2}$ [$=8-\frac{9n+16}{(n+1)(n+2)}=\frac{n(8n+15)}{(n+1)(n+2)}$]	M1 A1 M1 A1 ft M1 A1 ft A1 ft A1 **8**	Method for partial fractions Using partial fractions in at least two terms Three terms correct Cancelling to leave 3 fractions at beginning and/or 3 fractions at the end First 3 fractions correct Last 3 fractions correct (condone *r* instead of *n*)

3 (a)(i)	$\alpha^2 = 5 + 12\text{j}$ $\alpha^3 = (5 + 12\text{j})(3 + 2\text{j})$ $= -9 + 46\text{j}$	B1 M1 A1 **3**	or $\alpha^3 = 27 + 54\text{j} + 36\text{j}^2 + 8\text{j}^3$ Multiplication and use of $\text{j}^2 = -1$
(ii)(*A*)	$2(-9 + 46\text{j}) + p(5 + 12\text{j}) + 20(3 + 2\text{j}) + q = 0$ Real parts: $-18 + 5p + 60 + q = 0$ Imaginary parts: $92 + 12p + 40 = 0$ $p = -11,\ q = 13$	M1 A1 ft A1 ft M1 A1 **5**	Equating real or imaginary parts Solving to find *p* and *q*
(*B*)	Another root is $\alpha^* = 3 - 2\text{j}$	B1	
	$(x - \alpha)(x - \alpha^*) = x^2 - 6x + 13$ Equation is $(x^2 - 6x + 13)(2x + 1) = 0$	M1 M1	
	OR sum of roots $(3 + 2\text{j}) + (3 - 2\text{j}) + \beta = \frac{11}{2}$ M2		Correct equation (ft) *Condone a sign error*
	OR product of roots $(3 + 2\text{j})(3 - 2\text{j})\beta = -\frac{13}{2}$ M2		Correct equation (ft)
	Third root is $-\frac{1}{2}$	A1 **4**	$-\frac{1}{2}$ *always earns 3 marks*
(b)(i)	$\lvert\alpha\rvert = \sqrt{13}\quad (= 3.61)$ $\arg\alpha = \arctan\frac{2}{3} = 0.588$	B1 B1 **2**	Accept 3.6 Accept $\arctan\frac{2}{3}$, 0.59, 34°
(ii)		M1 A1 **2**	α in 1st quadrant, circle centre α correct circle (touching real axis) and inside of circle shaded
(iii)	(*A*) $\lvert\alpha\rvert - 2 = \sqrt{13} - 2\quad (= 1.61)$ (*B*) $\lvert\alpha\rvert + 2 = \sqrt{13} + 2\quad (= 5.61)$ (*C*) $2\arg\alpha = 1.18$	B1 ft B1 ft M1 A1 ft **4**	ft provided this is positive *SR* Give B1 if (*A*) & (*B*) interchanged angle identified (e.g. on diagram) Accept 67° *(Note: 68° earns M1A0)* *M0 if region is not a circle, or if circle touches y-axis, etc*

Alternatives for 3(a)(ii) (*A*) and (*B*) Total 9 marks

3(a)(ii)	Another root is $\alpha^* = 3 - 2\mathrm{j}$ $(x-\alpha)(x-\alpha^*) = x^2 - 6x + 13$ Equation is $(x^2 - 6x + 13)(2x + k) = 0$ Considering coefficients of x $-6k + 26 = 20$ $k = 1$ Third root is $-\frac{1}{2}$ Considering coefficients of x^2 $k - 12 = p$ Considering constant term, $13k = q$ $p = -11,\ q = 13$	B1 M1 M1 M1 A1 A1 M1 A1 ft A1		

3(a)(ii)	Another root is $\alpha^* = 3 - 2\mathrm{j}$ $\alpha\alpha^* + \alpha\gamma + \alpha^*\gamma = 10$ $13 + 6\gamma = 10$ Third root is $\gamma = -\frac{1}{2}$ $\alpha + \alpha^* + \gamma = -\dfrac{p}{2}$ $\alpha\alpha^*\gamma = -\dfrac{q}{2}$ $6 - \frac{1}{2} = -\dfrac{p}{2},\quad 13 \times (-\frac{1}{2}) = -\dfrac{q}{2}$ $p = -11,\ q = 13$	B1 M2 A1 A1 M1 M1 A1 ft A1		

4 (a)	When $n=1$, $\begin{pmatrix}1-3n & 9n\\ -n & 1+3n\end{pmatrix}=\begin{pmatrix}-2 & 9\\ -1 & 4\end{pmatrix}$	B1	
	Assuming it is true for $n=k$, $\mathbf{M}^{k+1}=\mathbf{M}^k\mathbf{M}=\begin{pmatrix}1-3k & 9k\\ -k & 1+3k\end{pmatrix}\begin{pmatrix}-2 & 9\\ -1 & 4\end{pmatrix}$	M1	or $\mathbf{MM}^k$ or $\begin{pmatrix}-2+6k-9k & -18k+9+27k\\ -1+3k-4k & -9k+4+12k\end{pmatrix}$
	$=\begin{pmatrix}-2+6k-9k & 9-27k+36k\\ 2k-1-3k & -9k+4+12k\end{pmatrix}$	A3	Give A2 for 2 elements correct, A1 for 1 element correct
	$=\begin{pmatrix}1-3(k+1) & 9(k+1)\\ -(k+1) & 1+3(k+1)\end{pmatrix}$	A1	Correctly obtained
	True for $n=k \Rightarrow$ True for $n=k+1$ Hence true for all positive integers n	A1 **7**	Stated or clearly implied *Dependent on previous 5 marks*
(b)(i)	$26\mathbf{i}-52\mathbf{j}+65\mathbf{k}$	B2 **2**	Give B1 for one coordinate correct *ISW e.g.* $=2\mathbf{i}-4\mathbf{j}+5\mathbf{k}$
(ii)	When $z=0$, $x=6$, $y=-4$	M1 A1	Finding one point on the line One point correct *or* $(0,8,-15)$ *or* $(4,0,-5)$
	Line is $\mathbf{r}=\begin{pmatrix}6\\-4\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\-4\\5\end{pmatrix}$	A1 ft **3**	ft on point and direction (if (i) not used, direction must be correct) *Condone omission of* $\mathbf{r}=$ Accept any form
	OR e.g. Eliminate z $(2x+y=8)$ and put $x=t$ M1 $y=8-2t$ A1 $z=-15+\frac{5}{2}t$ A1		
(iii)	Substituting into 3rd plane, $2(6+2\lambda)+(-4-4\lambda)+k(5\lambda)=13$	M1 A1 ft	
	$\lambda=\dfrac{1}{k}$	M1	Finding λ in terms of k
		M1	At least 2 of x, y, z in terms of k
	$x=6+\dfrac{2}{k}$, $y=-4-\dfrac{4}{k}$, $z=\dfrac{5}{k}$	A2 **6**	Give A1 for two correct
	OR e.g. Eliminating x in two ways, $5y+4z=-20$ $10y+(k+8)z=-35$ M2 M1 $x=6+2/k$, $y=-4-4/k$, $z=5/k$ A1A1A1		Finding one of x, y, z
	OR $\mathbf{M}^{-1}=\dfrac{1}{65k}\begin{pmatrix}10k-6 & 9k-8 & 26\\ -5k+12 & 2k+16 & -52\\ -15 & -20 & 65\end{pmatrix}$ M1 M1 A1		Attempt at cofactors Attempt at 1/det $\mathbf{M}$ Correct inverse matrix
	$\begin{pmatrix}x\\y\\z\end{pmatrix}=\mathbf{M}^{-1}\begin{pmatrix}48\\-10\\13\end{pmatrix}=\begin{pmatrix}6+2/k\\-4-4/k\\5/k\end{pmatrix}$ M1 A2		Finding one of x, y, z Give A1 for two correct
(iv)	When $k=0$ there are no solutions The 3 planes form the sides of a prism	M1 A1 **2**	*or* Line of intersection of 2 planes is parallel to 3rd plane *or* Planes intersect pairwise in three parallel lines

Examiner’s Report

2604: Pure Mathematics 4

General Comments

There were many very good scripts, with about 40% of candidates scoring 50 marks or more (out of 60). The great majority were able to demonstrate their ability across the specification, and only about 10% of candidates scored less than 30 marks. In general the presentation of work was commendable, and most candidates appeared to have sufficient time to complete the paper. Most candidates chose to answer questions 1, 2 and 3. Some attempted all four questions, but, while it is impressive to see a script with perfect answers to four questions, most of these would have been better advised to concentrate on producing carefully checked solutions to three questions.

Comments on Individual Questions

1) This question, on curve sketching, was the best answered, with half the attempts scoring 17 marks or more (out of 20).

(i) Most candidates gave all three asymptotes correctly. These were often written down (as requested) with little or no working, but there was some laborious dividing out to find the horizontal asymptote.

(ii) Almost all candidates applied the quotient rule correctly to find the derivative; the most common error was to obtain the negative of the correct derivative. The stationary points were usually found correctly; however, careless algebraic manipulation, such as sign errors and taking $(18x)(2x)$ as $36x$, very often led to an incorrect quadratic equation.

(iii) The standard of curve sketching was generally good. Common errors were omission of the left-hand branch (presumably because this did not show in the window of a graphical calculator), omitting to calculate the points of intersection with the x-axis, and failing to show a maximum on the right-hand branch.

(iv) The great majority of candidates knew exactly what to do here and reflected the negative sections of their curve, although the points on the x-axis were not always as sharp as they should have been.

(v) This was found quite challenging. Many understood what to do, but gave only one or two of the three possible values of k. Ranges of values such as $4.5 \le k \le 5$ were quite often given. A fair number of candidates tried to use the ideas of discriminants of quadratic equations; these attempts often involved a lot of algebra but rarely produced anything of value.

2) This question, on series and inequalities, was also very well answered, and about 20% of candidates scored full marks.

(a) Almost all candidates realised that they should multiply out $(2r-1)(4r+3)$ and apply the standard formulae, although $\sum(-3)$ was very often taken as -3 instead of $-3n$. Many errors were made in obtaining the final, factorised, answer.

(b) About half the candidates solved this inequality correctly. The commonest approaches were to multiply by $x^2(x-1)^2$ or to rearrange the inequality as $\frac{2x-1}{x(x-1)} < 0$. Some sketched the graphs of $y = \frac{x}{x-1}$ and $y = \frac{x-1}{x}$, and found their point of intersection. Common errors included giving the complement of the correct solution, and thinking that the inequality was equivalent to $x^2 < (x-1)^2$.

(c) The partial fractions were usually found without difficulty, although some sign errors and algebraic slips did occur, and some assumed the form $\frac{A}{r(r+1)} + \frac{B}{r+2}$. Most candidates understood the process of telescoping a sum and obtained the correct expression $7 - \frac{5}{2} + \frac{7}{2} - \frac{2}{n+1} - \frac{5}{n+1} - \frac{2}{n+2}$, although many had *r* in place of *n*. A surprisingly common error was to calculate $7 - \frac{5}{2} + \frac{7}{2}$ as $7 - 1 = 6$. The expected form for the final answer is $8 - \frac{7}{n+1} - \frac{2}{n+2}$. Many candidates went on to combine the fractions; such work was time-consuming (and often contained errors), but was ignored by the examiner.

3) This question, on complex numbers, was answered well, with half the attempts scoring 16 marks or more.

(a)(i) This was usually answered correctly.

(ii) Most candidates substituted α, α^2 and α^3 into the equation and equated real and imaginary parts to find *p* and *q*. The quadratic equation satisfied by α and α^* was then obtained, and used to find the linear factor $(2x+1)$. A fairly common error was to go from $132 + 12p = 0$ to $p = 11$, and the coefficient 2 of x^3 was sometimes ignored.
A variety of other methods were used. Some candidates proceeded as above to find *p* and *q*, then considered the product (or the sum) of the roots. Such attempts often ignored the 2 or contained sign errors. Others wrote the cubic as $(x^2 - 6x + 13)(2x + k)$, found *k* by considering the coefficient of *x*, then multiplied out to find *p* and *q*.

(b)(i) The modulus and argument were usually found correctly.

(ii) Most candidates drew the correct circle, although some omitted to shade the inside. However, many drew circles centred on the origin or on $(-3, -2)$, and some drew regions with straight line boundaries.

(iii) This was found difficult. The minimum and maximum values of the modulus were often given correctly (although this sometimes involved a great deal of algebra, finding the points of intersection of $y = \frac{2}{3}x$ with the circle), but the correct answer for the maximum argument was quite rare. A very common error was to assume that $1+2\text{j}$ and $5+2\text{j}$ (the complex numbers at the far left and right of the circle) gave the extreme values of the modulus, and that $1+2\text{j}$ gave the maximum argument.

4) This question, on matrices, induction and vectors, was much less popular than the others, and was attempted by only about one third of the candidates. It was also the worst answered question, with an average mark of about 12.

(a) The proof by induction was handled quite well. Most candidates realised that a matrix product was required, and this was usually carried out competently. There were, however, some who considered expressions such as $\mathbf{M}^k + \mathbf{M}^{k+1}$.

(b)(i) Most candidates found the vector product correctly, although there were many arithmetical slips.

(ii) Most candidates found one point on both planes, and used their answer to part (i) to give the direction. However, many made slips when solving the simultaneous equations leading to the point, and some gave the equation of the line as a single linear equation in *x*, *y* and *z*.

(iii) Candidates who used their answer to part (ii) and substituted into the third equation usually made substantial progress and often obtained the correct answer. Many of these could not cope with the algebra, which was often more complicated than necessary because the direction of the line had not been simplified.
However, most candidates did not use their previous work and started afresh with the three equations. The usual method was elimination, but this required a systematic approach and few candidates were successful. Some used the inverse matrix, and this was impressive when done correctly.

(iv) This part was often omitted. All possible ways of intersecting, such as a sheaf or a single point, and even parallel planes, were given, as well as the correct 'prism'.